Thus, on the basis of the three-dimensional magnetoelasticity equations, a correct two-dimensional theory of shells and plates of finite conductivity has been constructed. This theory allows us to solve magnetoelasticity problems for shells and plates having finite dimensions.

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ON THE LOSS OF STABILITY OF NONSYMMETRIC STRICTLY CONVEX THIN SHALLOW SHELLS

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Values of the upper critical buckling loads of nonsymmetric strictly convex elastic shallow shells are determined when the relative wall thickness parameter is sufficiently small. Simple relationships are derived from which the mentioned values can be found if the character of the loading, the shell geometry, and the method of fixing the edge are known. In passing, asymptotic expansions of the solutions permitting a computation of the stress-strain state of shell in the precritical stage are constructed for the appropriate boundary value problems. As an

illustration, asymptotic values of the upper critical pressures are found for ellipsoidal shells subjected to uniform external pressure and for different fundamental methods of fixing the edge. A number of problems on the buckling of strictly convex thin shells has been examined in [1].

1. Formulation of the problem. A nonlinear modification of the theory of "mean" bending of an elastic shallow shell subjected to a transverse load is considered [2, 3]

$$\varepsilon^2\Delta^2 w - [w - z, F] - q = 0, \qquad \varepsilon^2\Delta^2 F + {}^1/_2\,[w, w] - [z, w] = 0$$

$$\varepsilon^2 F_{xx} = \frac{1}{1-\nu^2}\left[v_y + z_{yy}w + \frac{1}{2}w_y^2 + \nu\left(u_x + z_{xx}w + \frac{1}{2}w_x^2\right)\right]$$

$$\varepsilon^2 F_{xy} = -\frac{1}{2(1+\nu)}\left[u_y + v_x + 2z_{xy}w + w_x w_y\right] \tag{1.1}$$

$$\varepsilon^2 F_{yy} = \frac{1}{1-\nu^2}\left[u_x + z_{xx}w + \frac{1}{2}w_x^2 + \nu\left(v_y + z_{yy}w + \frac{1}{2}w_y^2\right)\right]$$

$$\Delta w = w_{xx} + w_{yy}, \qquad [F, w] = F_{xx}w_{yy} + F_{yy}w_{xx} - 2F_{xy}w_{xy}$$

All the quantities in (1.1) are dimensionless and connected by the dimensional relationships

$$Z = az, \quad W = aw, \quad U = au, \; V = av, \quad x_1 = ax, \; y_1 = ay$$

$$\varepsilon^2 = h\,(a\gamma)^{-1}, \quad \Phi = Ea^2\varepsilon^2 F, \quad p = E\gamma\varepsilon^4 q, \quad \gamma^2 = 12\,(1-\nu^2)$$

Here Z is the shell middle surface, U, V, W are the displacements along the coordinate axes Ox_1, Oy_1, Oz_1, respectively, Φ is the Airy stress function, p is the external load intensity (pressure), and E is Young modulus. It is assumed that tne shell occupies a finite simply-connected convex domain D with boundary Γ, where the shell edge coincides with Γ, i.e. $z(s) = 0$ if $s \in \Gamma$. The small parameter ε^2 characterizes the relative wall thickness of the shell, h is the thickness, ν is the Poisson's ratio, and a is the characteristic dimension of the domain D. The deflection W is measured from the surface Z in the direction of load action.

Equations (1.1) are investigated, together with each of the boundary conditions on the contour Γ

$$\begin{aligned} &1)\; F = F_\rho = w = [w_{\rho\rho} + \nu\,(w_{ss} - \varkappa w_\rho)] = 0\\ &2)\; F = F_\rho = w = w_\rho = 0\\ &3)\; u = v = w = [w_{\rho\rho} + \nu\,(w_{ss} - \varkappa w_\rho)] = 0\\ &4)\; u = v = w = w_\rho = 0 \end{aligned} \tag{1.2}$$

Here $\varkappa = \varkappa(s)$ is the curvature of the contour Γ, where $\varkappa > 0$; s is the arclength parameter, and ρ is the interior normal to Γ. The boundary conditions (1.2) correspond to: (1) a moving hinged edge support, (2) sliding clamping of the edge, (3) a fixed hinged support, (4) absolutely rigid framing of the edge. Moreover, the surface $z(x, y)$ is assumed strictly convex, and the functions $q(x, y)$, $z(x, y)$ and $F_0(x, y)$ from (1.4) (see below) are sufficiently smooth.

Asymptotic values of the upper critical loads of arbitrary strictly convex shallow shells for the mentioned methods of edge fixing, when ε^2 tends to zero, are determined herein. In passing, asymptotic expansions of the solutions of the problems (1.1), (1.2) are constructed as $\varepsilon \to 0$. To do this, we use methods of asymptotic integration of the shell theory equations developed in [4 – 8].

Let $\varepsilon = 0$. We then have from (1.1)

$$\tfrac{1}{2}[w_0, w_0] - [z, w_0] = 0, \qquad [w_0 - z, F_0] + q = 0 \tag{1.3}$$

The former is the Monge-Ampère equation, and has two solutions under the boundary conditions $w_0(s) = 0$ such that

$$1)\ w_0 = 0, \quad [z, F_0] = q; \quad 2)\ w_0 = 2z, \quad [z, F_0] = -q \tag{1.4}$$

The first of these solutions corresponds, for small values of the parameter ε to the fundamental elastic equilibrium mode close to the initial surface z, since (1.1) are satisfied to the accuracy of quantities on the order of ε^2, but the boundary conditions (1.2) are hence not satisfied. It is shown below that as $\varepsilon \to 0$ the problems (1.1), (1.2) have solutions which behave similarly to (1) in (1.4) everywhere within the domain D and undergo strong changes near the boundary Γ such that the boundary conditions (1.2) are satisfied. These changes are described by the edge effect equations whose solution for arbitrarily assigned z and q reduces to integrating the edge effect equations for a spherical shell under uniform external pressure. These latter are solved numerically on an electronic digital computer in [8].

The Pogorelov [1] hypothesis that the strain of a sufficiently thin shell in the precritical stage is mainly an isometric transformation identical with the initial surface and the shell experiences substantial strain only in the neighborhood of the boundary of the buckling domain is confirmed here.

2. Strictly convex shells with free fixing of the edge. The asymptotic expansions of the solutions of problems (1.1), (1) and (2) in (1.2) are constructed as $\varepsilon \to 0$ in the neighborhood of the first solution in (1.4) as

$$F(x, y, \varepsilon) \sim \sum_{i=0}^{n} \varepsilon^i [F_i(x, y) + \varepsilon h_i(x, y, \varepsilon)] \tag{2.1}$$
$$w(x, y, \varepsilon) \sim \sum_{i=0}^{n} \varepsilon^i [w_i(x, y) + \varepsilon g_i(x, y, \varepsilon)]$$

The functions F_i, w_i are obtained by using the first iteration process [9]. Namely, letting $\mathbf{V} \equiv (F, w)$ denote the solution, and $\mathbf{P}(\mathbf{V})$ the left side of the system from the first two equations in (1.1), let us require that

$$\mathbf{P}(\mathbf{V}_n) = O(\varepsilon^{n+1}), \qquad \mathbf{V}_n \equiv \Big(\sum_{i=0}^{n} \varepsilon^i F_i, \sum_{i=0}^{n} \varepsilon^i w_i\Big) \tag{2.2}$$

Collecting coefficients of $\varepsilon^\circ, \varepsilon^1, \ldots \varepsilon^n$ and equating the expressions obtained to zero, we have for the determination of F_0, w_0

$$w_0 = 0, \qquad [z, F_0] = q, \qquad F_0|_\Gamma = A_0(s) \equiv 0 \tag{2.3}$$

and a system of linear, second-order, partial differential equations of elliptic type for the determination of F_i, w_i

$$[w_i, z] = \frac{1}{2} \sum_{k+j=i} [w_k, w_j] + \Delta^2 F_{i-2}$$
$$[F_i, z] = \sum_{k+j=i} [w_k, F_j] - \Delta^2 w_{i-2} \qquad (k, j \neq 0) \tag{2.4}$$
$$F_i|_\Gamma = A_i(s), \qquad w_i|_\Gamma = B_i(s) \qquad (i = 1, 2, \ldots n;\ F_{-1} = w_{-1} \equiv 0)$$

The right sides of (2.4) are known if $F_0, w_0, \ldots, F_{i-1}, w_{i-1}$ have already been found. The functions $A_i(s)$, $B_i(s)$ in the boundary conditions will be determined somewhat later in (2.11).

The vector V_n does not satisfy the boundary conditions (1) or (2) in (1.2) and the residuals originating are cancelled by functions of boundary layer type h_i, g_i which are determined by using the second iteration process [9]. To do this the difference $V - V_n$ is sought in the form (2.1). After substituting (2.1) into (1.1), we take account of (2.2) and go over to local coordinates (ρ, φ) of the boundary Γ in the relationships obtained by means of the formulas

$$\psi_x = \psi_\rho\rho_x + \psi_\varphi\varphi_x, \qquad \psi_y = \psi_\rho\rho_y + \psi_\varphi\varphi_y$$

We hence have

$$\sum_{i=0}^{n} \varepsilon^{i+3}\Delta^2 h_i + \frac{1}{2}\sum_{k,\, i=0}^{n} \varepsilon^{k+i+2}[g_k, g_i] + \sum_{k,\, i=0}^{n} \varepsilon^{k+i+1}[w_k, g_i] - \sum_{i=0}^{n} \varepsilon^{i+1}[g_i, z] = O(\varepsilon^{n+1})$$

$$\sum_{i=0}^{n} \varepsilon^{i+3}\Delta^2 g_i - \sum_{k,\, i=0}^{n} \varepsilon^{k+i+2}[g_k, h_i] + \sum_{i=0}^{n} \varepsilon^{i+1}[h_i, z] -$$

$$\sum_{k,\, i=0}^{n} \varepsilon^{k+i+1}[F_k, g_i] - \sum_{k,\, i=0}^{n} \varepsilon^{k+i+1}[w_k, h_i] = O(\varepsilon^{n+1}) \qquad (2.5)$$

Here

$$[u, \psi] = u_{xx}\psi_{yy} + u_{yy}\psi_{xx} - 2u_{xy}\psi_{xy}$$

$$\psi_{xy} = \psi_{\rho\rho}\rho_x\rho_y + \psi_{\rho\varphi}(\rho_x\varphi_y + \rho_y\varphi_x) + \psi_{\varphi\varphi}\varphi_x\varphi_y + \psi_\rho\rho_{xy} + \psi_\varphi\varphi_{xy}$$

$$\Delta^2\psi = \sum_{l=1}^{4} \sum_{m+k=l} \alpha_{mk}^{(l)} \frac{\partial^l \psi}{\partial\rho^m \partial\varphi^k}$$

Then we expand F_k, w_k, $\alpha_{mk}^{(l)}$, ρ_x, φ_x, ρ_{xx}, ρ_{xy}, $\ldots$ in a Taylor series in the neighborhood of $\rho = 0$, set $\rho = \varepsilon t$, collect coefficients of identical powers of ε, and derive equations to determine h_i, g_i by equating the expressions obtained for ε^{-2}, ε^{-1}, $\ldots$, ε^{n-1} to zero.

Let us note some valid relationships on the contour Γ. Sufficient smoothness of $\rho(x, y)$, $\varphi(x, y)$, as well as for the arbitrary function $\psi(x, y)$ relative to its arguments is hence assumed

$$\rho_x^2 + \rho_y^2 = 1, \qquad \rho_x = -Y_\varphi\delta^{-1}, \qquad \rho_y = X_\varphi\delta^{-1}$$

$$\varphi_x = X_\varphi\delta^{-2}, \qquad \varphi_y = Y_\varphi\delta^{-2}, \qquad \delta^2 = X_\varphi^2 + Y_\varphi^2$$

$$\rho_x^2\rho_{xx} + \rho_y^2\rho_{yy} + 2\rho_x\rho_y\rho_{xy} = 0, \qquad \rho_x^2\rho_{yy} + \rho_y^2\rho_{xx} - 2\rho_x\rho_y\rho_{xy} = -\varkappa(\varphi),$$

$$\psi_{xx}\rho_y^2 + \psi_{yy}\rho_x^2 - 2\psi_{xy}\rho_x\rho_y = \psi_{\varphi\varphi}\delta^{-2} - \psi_\varphi\delta^{-4}(X_\varphi X_{\varphi\varphi} + Y_\varphi Y_{\varphi\varphi}) - \varkappa(\varphi)\psi_\rho =$$

$$\psi_{ss} - \varkappa(s)\psi_\rho, \qquad \alpha_{40}^{(4)} = 1, \qquad \alpha_{31}^{(1)} = 0, \qquad \alpha_{30}^{(4)} = -2\varkappa \qquad (2.6)$$

Here $\varkappa = \varkappa(\varphi)$ is the curvature of the contour Γ at a point corresponding to the value of the parameter φ (or the arclength parameter s); $X = X(\varphi)$, $Y = Y(\varphi)$ are parametric equations of the curve Γ in the positive direction. Now, by using (2.6), we obtain that the ε^{-2} coefficient is identically zero. The ε^{-1} coefficient results in a system of nonlinear ordinary differential equations for the determination of h_0, g_0

$$\frac{\partial^4 h_0}{\partial t^4} - \frac{1}{2}\varkappa \frac{\partial}{\partial t}\left(\frac{\partial g_0}{\partial t}\right)^2 + \varkappa c \frac{\partial^2 g_0}{\partial t^2} = 0, \qquad \frac{\partial^4 g_0}{\partial t^4} +$$

$$\varkappa \frac{\partial}{\partial t}\left(\frac{\partial g_0}{\partial t}\frac{\partial h_0}{\partial t}\right) + f\varkappa \frac{\partial^2 g_0}{\partial t^2} - \varkappa c \frac{\partial^2 h_0}{\partial t^2} = 0 \tag{2.7}$$

$$f = F_{0\rho}|_\Gamma, \qquad c = z_\rho|_\Gamma > 0$$

In deriving the values for f and c it is taken into account that $F_0(s) = z(s) = 0$, if $s \in \Gamma$. We obtain a system of linear differential equations with variable coefficients of the form

$$\frac{\partial^4 h_i}{\partial t^4} - \varkappa \frac{\partial}{\partial t}\left(\frac{\partial g_0}{\partial t}\frac{\partial g_i}{\partial t}\right) + \varkappa c \frac{\partial^2 g_i}{\partial t^2} = R_{i1} \tag{2.8}$$

$$\frac{\partial^4 g_i}{\partial t^4} + \varkappa \frac{\partial}{\partial t}\left(\frac{\partial h_0}{\partial t}\frac{\partial g_i}{\partial t}\right) + \varkappa \frac{\partial}{\partial t}\left(\frac{\partial g_0}{\partial t}\frac{\partial h_i}{\partial t}\right) + f\varkappa \frac{\partial^2 g_i}{\partial t^2} - \varkappa c \frac{\partial^2 h_i}{\partial t^2} = R_{i2}$$

to determine h_i, g_i $(i \geqslant 1)$. Here R_{i1}, R_{i2} are known functions if $F_0, w_0, \ldots F_{i-1}, w_{i-1}; h_0, g_0, \ldots h_{i-1}, g_{i-1}$ have already been found.

Let us find the boundary conditions for h_i, g_i $(i \geqslant 0)$. To do this, let us substitute (2.1) into those boundary conditions (1.2) which contain the derivatives. Assuming $\rho = \varepsilon t$, and equating coefficients of identical powers of ε to zero, we obtain

$$1)\ \frac{\partial h_0}{\partial t}\bigg|_{t=0} = -F_{0\rho}|_\Gamma, \qquad \frac{\partial^2 g_0}{\partial t^2}\bigg|_{t=0} = 0, \qquad \frac{\partial h_i}{\partial t}\bigg|_{t=0} = -F_{i\rho}|_\Gamma$$

$$\frac{\partial^2 g_i}{\partial t^2}\bigg|_{t=0} = \nu\varkappa \frac{\partial g_{i-1}}{\partial t}\bigg|_{t=0} - \nu g_{i-2,ss}|_{t=0} - [w_{i-1,\rho\rho} + \nu w_{i-1,ss} - \nu\varkappa w_{i-1,\rho}]_\Gamma \tag{2.9}$$

$$2)\ \frac{\partial h_0}{\partial t}\bigg|_{t=0} = -F_{0\rho}|_\Gamma, \qquad \frac{\partial g_0}{\partial t}\bigg|_{t=0} = 0, \qquad \frac{\partial h_i}{\partial t}\bigg|_{t=0} = -F_{i\rho}|_\Gamma$$

$$\frac{\partial g_i}{\partial t}\bigg|_{t=0} = -w_{i\rho}|_\Gamma \qquad (i = 1, 2, \ldots n;\ g_{-1} \equiv 0)$$

Moreover, four more conditions result from the requirement that the functions h_i, g_i vanish at infinity

$$\left\{h_i, g_i, \frac{\partial h_i}{\partial t}, \frac{\partial g_i}{\partial t}\right\}_{t\to\infty} \to 0 \qquad (i = 0, 1, \ldots n) \tag{2.10}$$

Now, let us determine the boundary conditions for F_i, w_i. Satisfying (1.2) by using (2.1), we have

$$\left[F_0 + \sum_{i=1}^{n} \varepsilon^i (F_i + h_{i-1})\right]_\Gamma = O(\varepsilon^{n+1}), \qquad \left[w_0 + \sum_{i=1}^{n} \varepsilon^i (w_i + g_{i-1})\right]_\Gamma = O(\varepsilon^{n+1})$$

It hence follows that

$$F_0|_\Gamma = w_0|_\Gamma = 0, \quad A_i(s) = -h_{i-1}(0), \quad B_i(s) = -g_{i-1}(0) \tag{2.11}$$

The first relationship indicates the correctness of the selection of the boundary condition in (2.3), and the second permits predetermination of the problem (2.4).

Thus, the construction of asymptotics of the solutions of (1.1) under the boundary conditions (1), (2) in (1.2) reduces to the following. First F_0, w_0 are determined from (2.3), (2.11), and then h_0, g_0 from (2.7), (2.10). Furthermore, F_1, w_1 are determined from (2.4), (2.11), and then h_1, g_1 from (2.8) – (2.10), etc. Making the substitution

$$\frac{\partial h_0}{\partial t} = -\alpha c, \quad \frac{\partial g_0}{\partial t} = -\beta c, \quad \tau = \sqrt{\varkappa c}\, t, \quad \frac{1}{2} Q = f c^{-1}$$

we obtain from (2.7), (2.9), (2.10)

$$\frac{\partial^2 \alpha}{\partial \tau^2} + \frac{1}{2}\beta^2 + \beta = 0, \quad \frac{\partial^2 \beta}{\partial \tau^2} - \alpha\beta - \alpha + \frac{1}{2} Q\beta = 0, \quad \{\alpha, \beta\}_\infty \to 0 \quad (2.12)$$

with the corresponding boundary conditions

$$1)\ \alpha(0) = \frac{1}{2} Q, \quad \frac{\partial \beta}{\partial \tau}\Big|_{\tau=0} = 0; \quad 2)\ \alpha(0) = \frac{1}{2} Q, \quad \beta(0) = 0 \quad (2.13)$$

Therefore, for arbitrarily given z and q the solution of the equations for the main term in the edge effect zone reduces to the very same system (2.12). The problems (2.12), (2.13) have been solved numerically in [8]. The least branchpoints Q^* for these problems have also been found there. Then, let us introduce the quantity

$$\sigma = \max_s Q = \max_s [2fc^{-1}] = \max_s \left[\frac{2}{z_\rho(s)} \int_D G_\rho(x, y; \xi, \eta)\, q(\xi, \eta)\, d\xi\, d\eta\right]$$

$$f = F_{0\rho}(s), \quad c = z_\rho(s), \quad s \in \Gamma$$

as the load parameter [8]. Here G is the Green's function for the problem (2.3), and the point $(x, y) \in \Gamma$. Then using the result of Sect. 3 in [8], we obtain the respective asymptotic values of the upper critical load

$$1)\ \sigma_0 = Q^* = 0.793, \qquad 2)\ \sigma_0 = Q^* = 1.766 \quad (2.14)$$

for the boundary conditions (1) and (2) in (1.2). This value can be refined if we use series of perturbation theory

$$\sigma^* \sim \sum_{i=0}^{n} \varepsilon^i \sigma_i, \qquad q^*(x, y) \sim q(x, y) + \sum_{i=1}^{n} \varepsilon^i q_i \quad (2.15)$$

Here the q_i are constants determined together with σ_i from the condition that the linear boundary value problems of the second iteration process are solvable for $i \geqslant 1$. Thus, by passing to dimensional variables we arrive at the following result.

Let z, q and F_0 from (2.3) be sufficiently smooth functions in $D + \Gamma$. Then for very thin shells with the edge fixing conditions (1), (2) in (1.2), the values of the upper critical load P_j^* are determined by the formula

$$P_j^* = \frac{Eh^2}{\gamma a^2} \sigma_j^* = \max_{s \in \Gamma} \left[\frac{2}{z_\rho(s)} \int_D G_\rho(x, y; \xi, \eta)\, p_j^*(\xi, \eta)\, d\xi\, d\eta\right] =$$

$$\frac{\alpha_j E}{\sqrt{3(1-\nu^2)}} \left(\frac{h}{a}\right)^2 [1 + a_{1j}\varepsilon + \ldots], \quad j = 1, 2; \quad \alpha_1 = 0.3965, \quad \alpha_2 = 0.883 \quad (2.16)$$

Here the subscripts $j = 1,2$ correspond to the boundary conditions (1), (2) in (1.2), a is the characteristic dimension of the domain D and G is the Green's function of the problem (2.3). (The coefficients a_{ij} are not found herein).

3. Strictly convex shells under rigid edge fixing. The construction of the asymptotics in this case is rather more complex than in the case of free edge fixing. Indeed, even the determination of F_0 from (1.4) encounters difficulties in constructing the first iteration process (it is not known what boundary conditions are obtained for F_0 at $\varepsilon = 0$). Hence, equations connecting the function F with the variables u, v, w are relied upon in both iteration processes.

Equating the third order mixed derivatives for the function F, we obtain from (1.1)

$$L_1(u, v) \equiv v_{yy} + \frac{1-\nu}{2} v_{xx} + \frac{1+\nu}{2} u_{xy} = f_{11}(w) + f_{12}(w, w)$$

$$L_2(u, v) \equiv u_{xx} + \frac{1-\nu}{2} u_{yy} + \frac{1+\nu}{2} v_{xy} = f_{21}(w) + f_{22}(w, w)$$

$$f_{11}(w) = -[(z_{yy} + \nu z_{xx}) w]_y - (1-\nu)(z_{xy} w)_x$$

$$f_{21}(w) = -[(z_{xx} + \nu z_{yy}) w]_x - (1-\nu)(z_{xy} w)_y \tag{3.1}$$

$$f_{12}(w_1, w_2) = -w_{1y} w_{2yy} - \nu w_{1x} w_{2xy} - \frac{1-\nu}{2}(w_{1y} w_{2xx} + w_{1x} w_{2xy})$$

$$f_{22}(w_1, w_2) = -w_{1x} w_{2xx} - \nu w_{1y} w_{2xy} - \frac{1-\nu}{2}(w_{1y} w_{2xy} + w_{1x} w_{2xy})$$

Let us note that the relations (3.1), together with the first equation from (1.1) and the boundary conditions (3) or (4) in (1.2), comprise a complete system of equations for the mean bending of shells, written in terms of displacements [2, 3].

As before, the asymptotic expansions for F and w are constructed in the form (2.1), and for u and v in the form

$$u \sim \sum_{i=0}^{n} \varepsilon^i u_i + \sum_{i=0}^{n} \varepsilon^{i+1} \eta_i, \qquad v \sim \sum_{i=0}^{n} \varepsilon^i v_i + \sum_{i=0}^{n} \varepsilon^{i+1} \zeta_i \tag{3.2}$$

As a result of the first iteration process, we have (1.3) to determine F_0, w_0 from (1.1), and the following system for u_0, v_0 :

$$\begin{gathered} v_{0y} + z_{yy} w_0 + {}^1/_2\, w^2_{0y} + \nu\,(u_{0x} + z_{xx} w_0 + {}^1/_2\, w_{0x}{}^2) = 0 \\ u_{0x} + z_{xx} w_0 + {}^1/_2\, w_{0x}{}^2 + \nu\,(v_{0y} + z_{yy} w_0 + {}^1/_2\, w_{0y}{}^2) = 0 \\ u_{0y} + v_{0x} + 2 z_{xy} w_0 + w_{0x} w_{0y} = 0 \end{gathered} \tag{3.3}$$

We obtain (2.4) to determine F_i, w_i and the following system for u_i, v_i $(i \geqslant 1)$:

$$F_{i-2,\, xx} = \frac{1}{1-\nu^2}\left[v_{iy} + z_{yy} w_i + \frac{1}{2} \sum_{k+m=i} (w_{ky} w_{my} + \nu w_{kx} w_{mx}) + \nu u_{ix} + \nu z_{xx} w_i\right]$$

$$F_{i-2,\, xy} = \frac{-1}{2(1+\nu)}\left[u_{iy} + v_{ix} + 2 z_{xy} w_i + \sum_{k+m=i} w_{kx} w_{my}\right] \quad (F_{-1} \equiv 0) \tag{3.4}$$

$$F_{i-2,\, yy} = \frac{1}{1-\nu^2}\left[u_{ix} + z_{xx} w_i + \frac{1}{2} \sum_{k+m=i} (w_{kx} w_{mx} + \nu w_{ky} w_{my}) + \nu v_{iy} + \nu z_{yy} w_i\right]$$

Analogously to the derivation of (3.1) for u_0, v_0, we have from (3.3)

$$L_1(u_0, v_0) = f_{11}(w_0) + f_{12}(w_0, w_0), \qquad L_2(u_0, v_0) = f_{21}(w_0) + f_{22}(w_0, w_0) \tag{3.5}$$

and from (3.4) the following system for u_i, v_i :

$$L_1(u_i, v_i) = f_{11}(w_i) + \sum_{k+m=i} f_{12}(w_k, w_m) \tag{3.6}$$

$$L_2(u_i, v_i) = f_{21}(w_i) + \sum_{k+m=i} f_{22}(w_k, w_m)$$

Requiring that the expansions (2.1) and (3.2) be satisfied on the boundary Γ by the relationships (3) or (4) in (1.2), we obtain the following boundary conditions for the systems (2.4) and (3.6):

$$w_i|_\Gamma = -g_{i-1}|_\Gamma, \quad u_i|_\Gamma = -\eta_{i-1}|_\Gamma, \quad v_i|_\Gamma = -\zeta_{i-1}|_\Gamma \qquad (3.7)$$
$$(i = 0, 1, 2, \ldots; \quad g_{-1} = \eta_{-1} = \zeta_{-1} \equiv 0)$$

Thus, in order to determine w_i, u_i, v_i at each stage, it is necessary to know the value of the boundary-layer functions g_{i-1}, η_{i-1}, ζ_{i-1}, on the boundary Γ, which are determined as a result of the second iteration process. Let us show that

$$w_0 = u_0 = v_0 \equiv 0, \qquad w_1 = u_1 = v_1 \equiv 0 \qquad (3.8)$$

The first relationship is obtained directly from (3.7), (1.3) and (3.5). To prove the second relationship, let us first find the boundary condition (3.7) for $i = 1$. It will be shown below (see (3.17)) that $g_0 = \eta_0 = \zeta_0 = 0$ and therefore, $w_1(s) = u_1(s) = v_1(s) = 0$ for $s \in \Gamma$. Then (3.8) results from (2.4) and (3.6) for $i = 1$. Furthermore, to determine u_2, v_2, w_2, we have two equations from (3.6) for $i = 2$ and an equation from (1.4), which is written by using (3.4) for $i = 2$ as

$$z_{xx}(u_{2x} + z_{xx}w_2 + \nu v_{2y} + \nu z_{yy}w_2) + z_{yy}$$
$$(v_{2y} + z_{yy}w_2 + \nu u_{2x} + \nu z_{xx}w_2) = q(1 - \nu^2) \qquad (3.9)$$

The boundary conditions are hence determined from (3.7) for $i = 2$. Now, if u_2, v_2, w_2 have been found, the second derivatives of the function F_0 are calculated by means of (3.4) for $i = 2$. Let us note that u_2, v_2, w_2, F_0 are found simultaneously with the determination of the boundary layer functions h_1, g_1, ζ_1, η_1. The subsequent terms of the first iteration process are constructed in an analogous manner.

Let us turn to the second iteration process. In order to simplify the calculations significantly, let us first carry out the second iteration process for the functions h_i, g_i by temporarily assuming that the functions F_i, w_i are known. Then, as in Sect. 2, we obtain the system (2.7) to determine h_0, g_0. The boundary conditions for the functions g_i at $t = 0$ are obtained exactly as in (2.9). In order to obtain the boundary conditions for h_i, let us use the relationship for F on the contour Γ, which easily follows from (3) or (4) in (1.2) (see [2])

$$[F_{\rho\rho} - \nu F_{ss} + \varkappa\nu F_\rho]_\Gamma = 0 \qquad (3.10)$$

Using (2.1) and the substitution $\rho = \varepsilon t$, and equating the coefficients of identical powers of ε, we obtain from (3.10)

$$\left.\frac{\partial^2 h_0}{\partial t^2}\right|_{t=0} = 0, \qquad \left.\frac{\partial^2 h_i}{\partial t^2}\right|_{t=0} = -R(F_{i-1}) - \left[\varkappa\nu\frac{\partial h_{i-1}}{\partial t} - \nu\frac{\partial^2 h_{i-2}}{\partial s^2}\right]_{t=0} \qquad (3.11)$$

$$R(F_i) = [F_{i\rho\rho} - \nu F_{iss} + \varkappa\nu F_{i\rho}]_\Gamma \qquad (i = 1, 2, \ldots n; \ h_{-1} \equiv 0)$$

Now, it follows from (2.7), (2.9), (2.10) and (3.11) for any function F_0 that $h_0 = g_0 \equiv 0$. Then, from (2.5) we arrive at a linear system of ordinary differential coefficients with constant coefficients for h_1, g_1

$$\frac{\partial^4 h_1}{\partial t^4} - \varkappa c\frac{\partial^2 g_1}{\partial t^2} = 0, \qquad \frac{\partial^4 g_1}{\partial t^4} - \varkappa c\frac{\partial^2 h_1}{\partial t^2} + f_1\varkappa\frac{\partial^2 g_1}{\partial t^2} = 0 \qquad (3.12)$$

$$c = c(s) = z_\rho|_\Gamma, \qquad f_1 = f_1(s) = [F_{0xx}\rho_y^2 + F_{0yy}\rho_x^2 - 2F_{0xy}\rho_x\rho_y]_\Gamma$$

with boundary conditions corresponding to (3), (4) in (1.2)

$$3)\ \frac{\partial^2 g_1}{\partial t^2}\Big|_{t=0} = 0, \qquad \frac{\partial^2 h_1}{\partial t^2}\Big|_{t=0} = -R(F_0), \qquad \{h_1, g_1\}_\infty \to 0$$

$$4)\ \frac{\partial g_1}{\partial t}\Big|_{t=0} = 0, \qquad \frac{\partial^2 h_1}{\partial t^2}\Big|_{t=0} = -R(F_0), \qquad \{h_1, g_1\}_\infty \to 0 \tag{3.13}$$

The solutions of these problems are written down explicitly

$$3)\ \frac{\partial h_1}{\partial t} = \frac{B}{2ab}\left[a\left(1 - \frac{1}{2}Q\right)x^{(1)} + b\left(1 + \frac{1}{2}Q\right)y^{(1)}\right]$$

$$\frac{\partial g_1}{\partial t} = \frac{B}{2ab}[ax^{(1)} + by^{(1)}] \tag{3.14}$$

$$4)\ \frac{\partial h_1}{\partial t} = \frac{B}{b}\left[\frac{1}{4}Qx^{(1)} - 2aby^{(1)}\right], \qquad \frac{\partial g_1}{\partial t} = \frac{B}{b}x^{(1)}$$

Here

$$x^{(1)} = e^{-a\tau}\sin b\tau, \qquad y^{(1)} = e^{-a\tau}\cos b\tau, \qquad \tau = (\varkappa c)^{1/2}t$$

$$a = \left(\frac{4-Q}{8}\right)^{1/2}, \qquad b = \left(\frac{4-Q}{8}\right)^{1/2}, \qquad c = z_\rho|_\Gamma, \qquad Q = 2f_1c_1^{-1}$$

$$f_1 = [F_{0ss} - \varkappa F_{0\rho}]_\Gamma, \qquad c_1 = [z_{ss} - \varkappa z_\rho]_\Gamma$$

$$B = -(\varkappa c^3)^{1/2}[F_{0\rho\rho} - \nu F_{0ss} + \varkappa\nu F_{0\rho}]_\Gamma = -(\varkappa c^3)^{1/2}R(F_0)$$

The functions h_i, g_i $(i \geqslant 1)$ are determined from equations of the form (3.12), but inhomogenous. Formulas (3.14) are valid only for $Q < 4$.

Furthermore, let us turn to the construction of the boundary layer functions ζ_i, η_i $(i = 0, 1, 2, \ldots)$. To do this, let us substitute (2.1), (3.2) into (1.1), let us take account of (3.3), (3.4), and let us turn to the local (ρ, φ) coordinates in the expressions obtained. Together with (2.4) we obtain

$$\sum_{i=0}^{n}\varepsilon^{i+3}h_{i,xy} = -\frac{1}{2(1+\nu)}\Big\{\sum_{i=0}^{n}\varepsilon^{i+1}(\eta_{i\rho}\rho_y + \eta_{i\varphi}\varphi_y + \zeta_{i\rho}\rho_y + \zeta_{i\varphi}\varphi_y + z_{xy}g_i) +$$
$$\sum_{k+m=0}\varepsilon^{k+m+1}[w_{k,x}(g_{m\rho}\rho_y + g_{m\varphi}\varphi_y) + w_{k,y}(g_{m\rho}\rho_x + g_{m\varphi}\varphi_x) +$$
$$\varepsilon(g_{k\rho}\rho_x + g_{k\varphi}\varphi_x)(g_{m\rho}\rho_x + g_{m\varphi}\varphi_x)]\Big\} + O(\varepsilon^{n+1})$$

$$\sum_{i=0}^{n}\varepsilon^{i+3}h_{i,xx} = \frac{1}{1-\nu^2}[e_1 + \nu e_2] + O(\varepsilon^{n+1})$$

$$\sum_{i=0}^{n}\varepsilon^{i+3}h_{i,yy} = \frac{1}{1-\nu^2}[e_2 + \nu e_1] + O(\varepsilon^{n+1}) \tag{3.15}$$

Here

$$e_1 = \sum_{i=0}^{n}\varepsilon^{i+1}(\zeta_{i\rho}\rho_y + \zeta_{i\varphi}\varphi_y + z_{yy}g_i) + \sum_{k+m=0}\varepsilon^{k+m+1}\Big(w_{k,y} + \frac{\varepsilon}{2}g_{k\rho}\rho_y +$$
$$\frac{\varepsilon}{2}g_{k\varphi}\varphi_y\Big)(g_{m\rho}\rho_y + g_{m\varphi}\varphi_y)$$

$$e_2 = \sum_{i=0}^{n} \varepsilon^{i+1} (\eta_{i\rho}\rho_x + \eta_{i\varphi}\varphi_x + z_{xx}g_i) + \sum_{k+m=0} \varepsilon^{k+m+1}\Big(w_{k,x} + \frac{\varepsilon}{2} g_{k\rho}\rho_x + \frac{\varepsilon}{2} g_{k\varphi}\varphi_x\Big)(g_{m\rho}\rho_x + g_{m\varphi}\varphi_x)$$

$$h_{xy} = h_{\rho\rho}\rho_x\rho_y + h_{\rho\varphi}(\rho_x\varphi_y + \rho_y\varphi_x) + h_{\varphi\varphi}\varphi_x\varphi_y + h_\rho\rho_{xy} + h_\varphi\varphi_{xy}$$

Now, let us expand the known functions and their dertivatives in (3.15) in Taylor series in the neighborhood of $\rho = 0$, let us set $\rho = \varepsilon t$, let us collect coefficients of identical powers of ε and let us equate the expressions obtained for ε^0, ε^1, ..., ε^n to zero. We obtain the system

$$\begin{gathered}\zeta_{0t}\rho_x + \eta_{0t}\rho_y + g_{0t}(w_{0x}\rho_y + w_{0y}\rho_x) + g_{0t}^2\rho_x\rho_y = 0 \\ \zeta_{0t}\rho_y + \nu\eta_{0t}\rho_x + g_{0t}(w_{0y}\rho_y + \nu w_{0x}\rho_x) + {}^1/_2\, g_{0t}^2(\rho_y^2 + \nu\rho_x^2) = 0 \\ \nu\zeta_{0t}\rho_y + \eta_{0t}\rho_x + g_{0t}(w_{0x}\rho_x + \nu w_{0y}\rho_y) + {}^1/_2\, g_{0t}^2(\rho_x^2 + \nu\rho_y^2) = 0\end{gathered} \tag{3.16}$$

for the coefficient ε^0 to determine η_0, ζ_0. We will show that

$$g_0 = \eta_0 = \zeta_0 \equiv 0. \qquad \eta_1 = \zeta_1 = 0 \tag{3.17}$$

The first relationship follows from the fact, already proved, that $g_0 = 0$, from (3.16) and the conditions $\{\zeta_0, \eta_0\}_\infty \to 0$. Then, equating the coefficient of ε^1 in (3.15) to zero, we have

$$\rho_x^2 h_{0tt} = \frac{1}{1-\nu^2}(\zeta_{1t}\rho_y + \nu\eta_{1t}\rho_x), \qquad \rho_y^2 h_{0tt} = \frac{1}{1-\nu^2}(\eta_{1t}\rho_x + \nu\zeta_{1t}\rho_y),$$

$$-\rho_x\rho_y h_{0tt} = \frac{1}{2(1+\nu)}(\zeta_{1t}\rho_x + \eta_{1t}\rho_y)$$

to prove the second relationship. Hence, (3.17) follows because $h_0 = 0$ and $\{\zeta_1, \eta_1\}_\infty \to 0$. By using (3.7), we determine the boundary conditions for the system (3.6) from the second relationship in (3.17) for $i = 2$; they are $u_2(s) = v_2(s) = 0$ for $s \in \Gamma$. Then u_2, v_2, w_2, are found from (3.6) for $i = 2$ and (3.9), and the second derivatives of the function F_0 from (3.4) for $i = 2$. Furthermore, equating the coefficient for ε^2 in (3.15) to zero and taking account of (3.17), we obtain the following system of linear equations to determine η_2, ζ_2:

$$\rho_x^2 h_{1tt} = \frac{1}{1-\nu^2}\Big[\zeta_{2t}\rho_y + z_{yy}g_1 + \nu\eta_{2t}\rho_x + \nu g_1 z_{xx} + \frac{1}{2} g_{1t}^2(\rho_y^2 + \nu\rho_x^2)\Big]$$

$$\rho_y^2 h_{1tt} = \frac{1}{1-\nu^2}\Big[\eta_{2t}\rho_x + z_{xx}g_1 + \nu\zeta_{2t}\rho_y + \nu g_1 z_{yy} + \frac{1}{2} g_{1t}^2(\rho_x^2 + \nu\rho_y^2)\Big]$$

$$-\rho_x\rho_y h_{1tt} = \frac{1}{2(1+\nu)}[\zeta_{2t}\rho_x + \eta_{2t}\rho_y + 2z_{xy}g_1 + g_{1t}^2\rho_x\rho_y]$$

with the boundary conditions $\{\zeta_2, \eta_2\}_\infty \to 0$. Here h_1 and g_1 are already known from (3.14). The boundary layer functions ζ_i, η_i $(i > 2)$ are determined analogously. Let us note that the formulas of the Pogorelov geometric method for strictly convex shells (see [1], ch. V) can be derived from (3.12) and (3.16).

Let us introduce the load parameter [8] defined by the formula

$$\sigma = \max_{s\in\Gamma} Q = \max_{s\in\Gamma}[2f_1c_1^{-1}] = \max_{s\in\Gamma}[2c_1^{-1}Lq] \tag{3.18}$$

$$f_1 = [F_{0xx}\rho_y^2 + F_{0yy}\rho_x^2 - 2F_{0xy}\rho_x\rho_y]_\Gamma, \qquad c_1 = -\varkappa(s)\, z_\rho(s)$$

Here L is a linear operator defined by the relationship $f_1 = Lq$. Furthermore, let us note that (3.14) are valid only for $Q < 4$. For $Q = 4$ the problems (3.12), (3.13) have no solutions which decrease at infinity. Repeating the same reasoning as in Sect. 4 of [8], we obtain the asymptotic value of the upper critical load in the case of the boundary conditions (3) and (4) in (1.2)

$$\sigma_0 = Q^* = 4 \tag{3.19}$$

Successive terms of the expansion in powers of ε for values of the upper critical load can be constructed by using the relationships (2.15). Thus, by returning to dimensional variables we arrive at the following result.

Let z, q and F_0 be sufficiently smooth functions in $D + \Gamma$. Then values of the upper critical load are determined for very thin shells with the edge fixing conditions (3), (4) in (1.2) by the formula

$$P_j^* = \frac{Eh^2}{\gamma a^2}\sigma_j^* = \max_{s\in\Gamma}\,[2c_1^{-1}Lp^*] =$$

$$\frac{2E}{\sqrt{3(1-\nu^2)}}\left(\frac{h}{a}\right)^2 [1 + a_{1j}\varepsilon + a_{2j}\varepsilon^2 + \ldots], \qquad j = 3, 4$$

Here the subscripts 3, 4 correspond to the boundary conditions (3), (4) in (1.2), a is the characteristic dimension of the domain D, and L is the linear operator defined by $f_1 = Lq$ (the coefficients a_{ij} are not found herein).

4. Ellipsoidal shell under uniform external pressure. Let the initial shell middle surface and the parametric equations of the contour Γ be given as

$$z = 1 - \frac{1}{2}(k_1x^2 + k_2y^2), \qquad X = \sqrt{\frac{2}{k_1}}\cos\varphi \tag{4.1}$$

$$Y = \sqrt{\frac{2}{k_2}}\sin\varphi, \quad k_1 = \frac{a}{R_1} > 0, \quad k_2 = \frac{a}{R_2} > 0, \quad z|_\Gamma = 0$$

In the case of the boundary conditions (1), (2) in (1.2), we find

$$k_2F_{0xx} + k_1F_{0yy} + q = 0, \qquad F_0|_\Gamma = 0$$

from (2.3) for the determination of F_0. It is easy to guess the solution of this problem. We hence have

$$F_0 = \frac{1}{4}\frac{q}{k_1k_2}(2 - k_1x^2 - k_2y^2)$$

Using (2.6), we deduce

$$f = F_{0\rho}|_\Gamma = {}^1/_2 q(\varkappa k_1k_2)^{-1}[k_1\rho_y^2 + k_2\rho_x^2]_\Gamma, \qquad c = z_\rho|_\Gamma =$$
$$\varkappa^{-1}[k_1\rho_y^2 + k_2\rho_x^2]_\Gamma, \qquad \sigma = q/k_1k_2$$

Then by using (2.14), we have for the cases (1), (2) in (1.2)

$$1)\ \sigma_0 = q_0/k_1k_2 = 0.793, \qquad 2)\ \sigma_0 = q_0/k_1k_2 = 1.766$$

Hence, returning to dimensional variables for the asymptotic value of the upper critical pressure, we correspondingly obtain

$$1)\ p_0 = \frac{0.3965}{\sqrt{3(1-\nu^2)}}\frac{h^2}{R_1R_2}, \qquad 2)\ p_0 = \frac{0.883}{\sqrt{3(1-\nu^2)}}\frac{h^2}{R_1R_2} \tag{4.2}$$

In the case of boundary conditions (3), (4) in (1.2), we use the formulas of Sect. 3 to determine F_0. Applying (4.1), we obtain from (3.9)

$$w_2 = [q(1-\nu^2) + (k_2+\nu k_1)v_{2y} + (k_1+\nu k_2)u_{2x}](k_1^2+k_2^2+2\nu k_1k_2)^{-1}$$

Substituting w_2 in (3.6) for $i = 2$ and taking account of the boundary conditions $v_2(s) = u_2(s) = 0$ $(s \in \Gamma)$, we deduce $v_2(x, y) = u_2(x, y) = 0$. Then for $i = 2$ it follows from (3.4):

$$w_2 = q(1-\nu^2)K, \qquad F_{0xx} = -q(k_2+\nu k_1)K$$

$$F_{0yy} = -q(k+\nu k_2)K, \qquad F_{0xy} = 0, \qquad K = (k_1^2+k_2^2+2\nu k_1k_2)^{-1} \tag{4.3}$$

Using (4.3) and also the relationship

$$[\psi_{\rho\rho} - \nu\psi_{ss} + \varkappa\nu\psi_\rho]_\Gamma = [(\psi_{xx}-\nu\psi_{yy})\rho_x^2 + (\psi_{yy}-\nu\psi_{xx})\rho_y^2 + 2(1+\nu)\psi_{xy}\rho_x\rho_y]_\Gamma$$

we obtain

$$R(F_0) = -q(1-\nu^2)K[k_2\rho_x^2 + k_1\rho_y^2]_\Gamma, \qquad c_1 = -[k_1\rho_y^2 + k_2\rho_x^2]_\Gamma \tag{4.4}$$

$$f_1 = -qK[\rho_y^2(k_2+\nu k_1) + \rho_x^2(k_1+\nu k_2)]_\Gamma$$

Now h_1, g_1 are determined completely by (3.14). In particular, we have in both cases (3) and (4)

$$g_1(0) = -q(1-\nu^2)K = w_2(s) \qquad (s \in \Gamma)$$

i.e. the first condition in (3.7) is sarisfied for $i = 2$. Furthermore, taking account of (2.6) and (4.4), in conformity with (3.18), we determine the load parameter

$$\sigma = \max_{0\leqslant\varphi\leqslant 2\pi}\left[\frac{2qK}{k_1k_2}(k_2^2\sin^2\varphi + k_1^2\cos^2\varphi + \nu k_1k_2)\right] = \frac{2q(1+\nu\alpha)}{k_1k_2(1+2\nu\alpha+\alpha^2)}$$

$$\alpha = k_1/k_2, \quad \text{if} \quad k_2 > k_1 \quad \text{and} \quad \alpha = k_2/k_1, \quad \text{if} \quad k_1 > k_2$$

Then by using (3.19) we have

$$\sigma_0 = \frac{2q_0(1+\nu\alpha)}{k_1k_2(1+2\nu\alpha+\alpha^2)} = 4, \qquad \alpha = \frac{\min(R_1, R_2)}{\max(R_1, R_2)} \leqslant 1$$

Hence, by passing to dimensionless variables, we obtain for the asymptotic value of the upper critical pressure in the case of boundary conditions (3) and (4) in (1.2)

$$p_0 = \frac{AE}{\sqrt{3(1-\nu^2)}}\frac{h^2}{R_1R_2}, \qquad A = \frac{1+2\nu\alpha+\alpha^2}{1+\nu\alpha} \tag{4.5}$$

Thus, for sufficiently thin elastic ellipsoidal shells under uniform external pressure and the boundary conditions (1.2), the values of the upper critical pressure are determined by the formula

$$p_j^* = \frac{\alpha_jE}{\sqrt{3(1-\nu^2)}}\frac{h^2}{R_1R_2}[1 + a_{1j}\varepsilon + a_{2j}\varepsilon^2 + \ldots] \tag{4.6}$$

$$j = 1, 2, 3, 4, \quad \alpha_1 = 0.3965, \quad \alpha_2 = 0.883$$

$$\alpha_3 = \alpha_4 = (1+2\nu\alpha+\alpha^2)(1+\nu\alpha)^{-1}, \qquad \alpha = \frac{\min(R_1, R_2)}{\max(R_1, R_2)}$$

Here the subscripts $j = 1$ to 4 correspond to the boundary conditions (1) – (4) of (1.2). (The coefficients a_{1j}, a_{2j}, ... are not found herein).

If the conditions $F|_\Gamma = 0$, $e_n|_\Gamma = 0$ (e_n is the displacement normal to the contour

Γ) are taken in (3), (4) from (1.2) instead of the conditions $u(s) = v(s) = 0$, then (4.5) is obtained for p_0, where $A = 2$. In this case the value of p_0 has been found earlier by Pogorelov by geometric methods (see [1]), and the influence of imperfections in fixing the edge on p_0 has been investigated in [10].

The appropriate expansions (2.1) and (3.2) describe the asymptotic behavior of an ellipsoidal shell in the precritical stage. The shell is mainly deformed as a rigid body and a strong change in the stresses, moments, etc. is observed only near the edge. The process of shell snapping starts in the edge effect zone, where this occurs at once along the whole support contour in the case of boundary conditions (1), (2) in (1.2), and starts with the formation of crescent-shaped dents in the neighborhood of the vertices of the minor axis of the ellipse D:

Setting $R_1 = R_2 = R$, we obtain the asymptotic value of the upper critical pressure of a spherical shell from (4.6)

$$p_j^* = \frac{\alpha_j E}{\sqrt{3(1-\nu^2)}}\left(\frac{h}{R}\right)^2 [1 + a_{1j}\varepsilon + a_{2j}\varepsilon^2 + \ldots] \qquad (4.7)$$

$$\alpha_1 = 0.3965, \qquad \alpha_2 = 0.883, \qquad \alpha_3 = \alpha_4 = 2$$

i.e. it agrees with the values found by axisymmetric theory [8].

It has been shown in [11 – 14] that the buckling of a thin spherical shell under uniform external pressure can occur in a nonsymmetric mode, where the number of harmonics n corresponding to the minimal critical load increases as the value of the parameter ε diminishes. Formula (4.7) results in the deduction that sufficiently thin spherical shells ($\varepsilon \to 0$) under uniform external pressure buckle in an axisymmetric mode (*).

However, for $j = 4$ formula (4.7) contradicts the asymptotic value of p_H^*, the upper critical pressure for buckling in a nonsymmetric mode, found in [11]

$$p_H^* = 0.864 p_4^*, \qquad \text{if} \qquad \frac{n^2 R h}{a^2 \sqrt{12(1-\nu^2)}} \to (0.817)^2 \qquad (4.8)$$

Here, the following must be noted. All the deductions herein (including (4.7)) have been obtained under the assumption that for $\varepsilon \to 0$ changes in the solutions in a direction of the normal to contour Γ (in the boundary layer) have a higher order of magnitude in ε^{-1} than along Γ. Moreover, it is easy to show that (4.7) holds for the cases $j = 3,4$ under the condition

$$\varepsilon^{2-\alpha} n^2 = O(1), \; 0 < \alpha \leqslant 2 \; (\varepsilon \to 0, \; n \to \infty).$$

As regards the result (4.8), it seems to be doubtful. The method of obtaining it contains many unfounded assumptions (for example, the boundary layer functions for the radial stress resultants in the axisymmetric solution are discarded).

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*) A.V. Pogorelov: Report to the All-Union Conference on Plate and Shell Theory, Rostov-on-Don, 1971.

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